Analysis of the mechanical vibrations of a complex structure as a damped single degree of freedom system

Análisis de vibraciones mecánicas de una estructura compleja como un sistema amortiguado de un grado de libertad

Análise de vibrações mecânicas de uma estrutura complexa como um sistema amortecido de um grau de liberdade

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Summary. - The use of finite element method software allows the modal and spectral analysis of complex structures with an accessible computational cost. In contrast, with the theory of damped systems of a degree of freedom, solutions can be obtained analytically that describe with greater generality the vibrations of structures subjected to variable forces over time. However, with analytical methods, only very simple structures can be studied. This paper presents a method that allows to calculate the rigidity and effective mass of a structure from the values of the angular frequencies of the structure, with their corresponding inertial loads. Next, the structure can be analyzed as a cushioned system of a degree of freedom. In this way, it is possible to calculate the displacements and accelerations that the structure will suffer when it is excited by an external force variable over time. Subsequently, using D'Alembert's principle and a finite element program, the stresses can be calculated by a static analysis.

Keywords: mechanical vibrations; modal analysis; damped single degree of freedom system.

Resumen. *- El uso de programas basados en el método de elementos finitos permite en la actualidad el análisis modal y el espectral de estructuras complejas con un costo computacional accesible. En contraposición, con la teoría de sistemas amortiguados de un grado de libertad se puede obtener analíticamente soluciones que describen con mayor generalidad las vibraciones de estructuras sometidas a fuerzas variables en el tiempo. Sin embargo, con métodos analíticos, sólo pueden ser estudiadas estructuras muy sencillas. En este trabajo se presenta un método que permite calcular la rigidez y la masa efectiva de una estructura a partir de los valores de las frecuencias angulares propias de la estructura, con sus correspondientes cargas inerciales. A continuación, la estructura puede ser analizada como un sistema amortiguado de un grado de libertad. De esta forma es posible calcular los desplazamientos y aceleraciones que sufrirá la estructura cuando es excitada por una fuerza externa variable con el tiempo. Posteriormente, utilizando el principio de D´Alembert y un programa de elementos finitos, se pueden calcular las solicitaciones con un análisis estático.*

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Palabras clave: *vibraciones mecánicas; análisis modal; sistema amortiguado de un grado de libertad.*

Resumo. **-** *A utilização de programas baseados no método dos elementos finitos permite atualmente a análise modal e espectral de estruturas complexas com um custo computacional acessível. Em contraste, com a teoria dos sistemas amortecidos de um grau de liberdade, é possível obter analiticamente soluções que descrevem de forma mais geral as vibrações de estruturas sujeitas a forças que variam no tempo. No entanto, com métodos analíticos, apenas estruturas muito simples podem ser estudadas. Este trabalho apresenta um método que permite o cálculo da rigidez e massa efetiva de uma estrutura a partir dos valores das frequências angulares da estrutura, com suas correspondentes cargas inerciais. A estrutura pode então ser analisada como um sistema amortecido de um grau de liberdade. Desta forma é possível calcular os deslocamentos e acelerações que a estrutura sofrerá quando for excitada por uma força externa que varia ao longo do tempo. Posteriormente, usando o princípio D'Alembert e um programa de elementos finitos, as forças podem ser calculadas com uma análise estática.*

Palavras-chave: *vibrações mecânicas; análise de modos; Sistema amortecido com um grau de liberdade.*

Nomenclature

ω_n	Natural angular frequency
ξ	Damping ratio
a	Slope of the function f
b	Independent variable of the function f
r_{eff}	Ratio of the structural mass
c	Damping coefficient
$f(t)$	Force applied to the load mass as time function
m_{eff}	Effective mass
m_{effA}	Effective mass analytically obtained
m_{effN}	Effective mass numerically obtained
m_L	Loaded mass
m_s	Structural mass
k	Rigidity of the structure

1. Introduction. -

In many applications, structures are subjected to time-dependent external forces within a broad range of frequencies. Therefore, large resonance effects may take place if an oscillation mode is excited [1]. Usual examples include winds on high voltage transmission lines, seismic loads in buildings, drive train vibrations in vehicles and structures with unbalance rigid rotors. This often results in degraded performance, and in severe cases, even requires costly repairs and/or modifications. Consequently, significant benefits result from a careful analysis at the design phase. Modal analysis by means of finite element methods enables the calculation of resonance frequencies of complex structures. Within the linear elastic range, present-day software packages and personal computers make it possible to achieve highly accurate results at reasonable cost. Even though real-world examples may have many vibration modes, in most cases the lowest frequency ("fundamental") mode is the most important from an engineering viewpoint. The fact that the response of a dynamic system up to the lowest resonance frequency may be modelled as a second-order system, is a well-known result, arrived by analytical mechanics. In other words, the response of the structure in the fundamental mode (see fig. I) is analogous to a damped single degree of freedom (i.e. mass-dashpot-spring) system, driven by an external time-varying force (see fig. II). The equivalent inertia of the structure is modelled by the effective mass, m_{eff}, and its elastic property is modelled by the rigidity, k. In most structures, the effective mass, m_{eff}, is a fraction of the total mass of the structure, ms, since, under the action of the external force f(t), the various parts of the structure have different accelerations. The mechanical energy dissipation in the structure (usually very low) is modelled by the damping coefficient c.

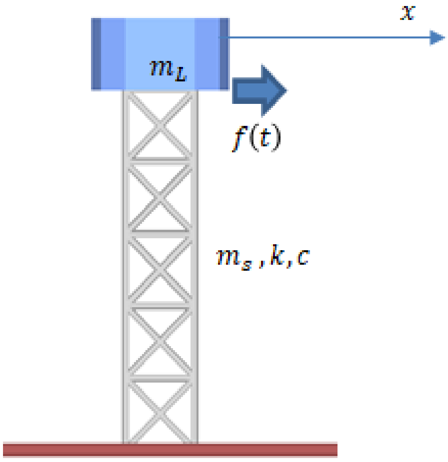

Figure I.- Structure with a transient force.

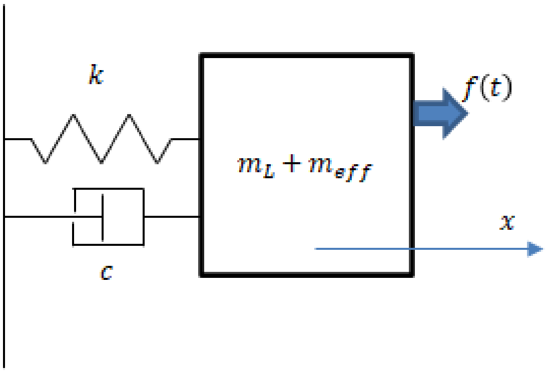

Figure II.- Damped single degree of freedom system

The equation of motion for the damped single degree of freedom system is:

$$\frac{k}{m_{eff} + m_L} x(t) + \frac{c}{m_{eff} + m_L} \frac{dx}{dt} + \frac{d^2 x}{dt^2} = \frac{f(t)}{m_{eff} + m_L}. \quad (1)$$

The solutions of eq. (1) when $f(t) = 0$ describe the free response of the system when displaced from its equilibrium configuration. For eq. (1) it is convenient to define the natural angular frequency, ω_n , corresponding to the free oscillation of the system with no damping (c = 0).

$$\omega_n = \sqrt{\frac{k}{m_{eff} + m_L}}. \quad (2)$$

The (dimensionless) damping ratio, ξ , proportional to the damping coefficient c, describes the decay of the free response:

$$2\xi\omega_n = \frac{c}{m_{eff} + m_L}. \quad (3)$$

From a dimensional analysis of eq. (1), it follows that the time scale of the decay of the free response is of the order of $(2\xi\omega_n)^{-1} = \left(\frac{c}{m_{eff}+m_L}\right)^{-1}$.

By substituting eqs. (2) and (3), in eq. (1), the equation of motion may be written as:

$$\omega_n{}^2 x(t) + 2\zeta\omega_n \frac{dx}{dt} + \frac{d^2 x}{dt^2} = \frac{f(t)}{m_{eff} + m_L}, \quad (4)$$

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Usually, the structural damping is described as a constant value. Bachman has shown usual values for several kinds of mechanical vibrations problems [5]. Radoičić and Jovanović [6] have shown an experimental and theoretical procedure for identifying values of the structural damping coefficient as well as a model for calculating the coefficient. They show an application example for a real tower crane structure. More complex models of the structural damping can be found on [7], including the well-known Rayleigh model,

$$\xi = \frac{1}{2\omega_n} a_0 + \frac{\omega_n}{2} a_1 , \tag{5}$$

where a_0 and a_1 are calculated from the structure geometry and material.

The D'Alembert's Principle consist in add to the external forces a fictitious force that, commonly known as inertial force to set the dynamic equilibrium [2]. The principle is power tool because it allows the usage of a static analysis software to solve a dynamic problem. The inertial force is given by

$$f_i(t) = -m_L \frac{d^2x}{dt^2} \tag{6}$$

Consequently, in the vibrating structure the structural mass has a differential inertial mass:

$$df_i(t) = -dm_s \frac{d^2x}{dt^2} \tag{7}$$

In the FEA software, the inertial force of the structure mass is applied in every node with eq. (7) and the loaded mass with eq. (6).

1.1. Case 1: Harmonic excitation. -

If the structure is subjected to an external excitation $f(t)$ with a harmonic time dependence,

$$f(t) = F_0 \sin(\omega_f t) , \tag{8}$$

the solution of eq. (4) gives the transient displacement [2], giving by.

$$x(t) = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega_f t + \theta) , \tag{9}$$

where

$$r = \frac{\omega_f}{\omega_n} \tag{10}$$

$$tg(\theta) = \frac{2\xi r}{1 - r^2} \tag{11}$$

1.2. Case 2: Instant force. -

In the case when the force, F, is applied instantly the solution of eq. (4) is

$$x(t) = \frac{F}{k}\sin(\omega_n t + \theta) \tag{12}$$

And the acceleration

$$\frac{d^2x}{dt^2} = -\omega_n^2 \frac{F}{k}\sin(\omega_n t + \theta) \tag{13}$$

2. Methodology. -

As indicated in the previous paragraphs, the rigidity and mass parameters that model the response of a structure (Figs. I and II) may be obtained from the natural frequency and the response to a static external force [2]. In most structures, the damping ratio, ξ, is very low. Therefore, analytical techniques may be used for simple geometries. For instance, the effective mass of the structure, m_{eff}, can be calculated using the Rayleigh method or, for beams, by the resolution of the Euler equation [2] [3]. However, for present-day engineering problems, numerical techniques are clearly necessary.

The effective mass of the structure is a fraction of the mass of the structure.

$$r_{eff} = \frac{m_s}{m_{eff}} \tag{14}$$

The function: $f(m_L) = {\omega_n}^{-2}$ can be defined from eq. (2). It is a linear function of m_L, where the slope is the inverse of the rigidity, and the constant is the ratio of the effective mass to the rigidity. Even though the f is also function of m_{eff} here is presented as a m_L function to obtain m_{eff}:

$$f(m_L) = {\omega_n}^{-2} = \frac{1}{k}m_L + \frac{m_{eff}}{k} \tag{15}$$

The natural frequencies of the system versus m_L can be found with a modal analysis, using a finite element software, with no limits on the complexity of the structure. Moreover, the natural frequencies could be also measured, for several values of m_L, by measuring the oscillation period for each case. The slope and the constant of the function *f*, eq. (15), can then be obtained with the approximation of the linear last squares [4], as shown in Fig. III.

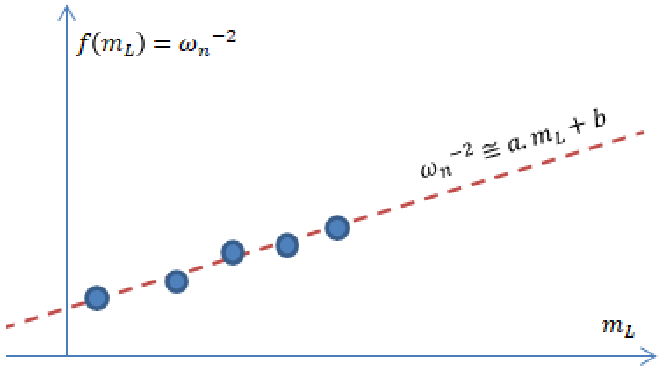

Figure III.- Schematic representation of the f function when the loaded mass and the rigidity are constant.

The slope and the constant of the function f, are given by:

$$a = \left(j \sum_{i=1}^{j} \omega_i{}^{-2} m_i - \sum_{i=1}^{j} m_i \sum_{i=1}^{j} \omega_i{}^{-2} \right) \Bigg/ \left(j \left(\sum_{i=1}^{j} m_i{}^2 \right) - \left(\sum_{i=1}^{j} m_i \right)^2 \right), \quad (16)$$

$$b = \left(\sum_{i=1}^{j} m_i{}^2 \sum_{i=1}^{j} \omega_i{}^{-2} - \sum_{i=1}^{j} \omega_i{}^{-2} m_i \sum_{i=1}^{j} m_i \right) \Bigg/ \left(j \left(\sum_{i=1}^{j} m_i{}^2 \right) - \left(\sum_{i=1}^{j} m_i \right)^2 \right). \quad (17)$$

Finally, the rigidity of the structure and its effective mass are given by

$$k = \frac{1}{a} \quad (18)$$

$$m_{eff} = b.k \quad (19)$$

3. Numerical examples. - Three examples are given in this Section and solved by the method of Section 2. The results of the first and second examples are compared to the results obtained by the Raleigh method. The third example shows the applicability of the method for a structure that, due to its complexity, cannot be analytically solved. In these three examples the natural frequencies versus m_L have been obtained with the software LISA 8, using the free version (1300 node limit). The properties of the structure material are given in Table I.

Mechanical properties	
Young Module	210 (Gpa)
Poisson coef.	0,3
Density	7800 (kg/m^3)

Table I.- Numerical values of the mechanical properties of the material used in the examples.

3.1. Example 1.- Cantilever beam

This example describes a cantilever beam of rectangular cross section (width10mm, height 20mm) and a length of 500 mm. The weight of a mass of 0,3 kg is applied instantly.
The model of the structure in LISA 8, with 4 beam elements and 5 nodes, has a fixed support on one end (See fig. IV). Seven runs were made with the added mass m_L of 0.2, 0.4, 0.8, 1.0, 2.0, 3.0 and 4 kg on the other end. A screenshot with the result of one run is shown on figure V.

Table II shows the results for all the values of m_L, and the values of the function $f(m_L)$ are plotted in fig. VI. The calculated slope is 2,975.10^{-7}[N/m], and the constant is 5,508.10^{-6} [s^2].

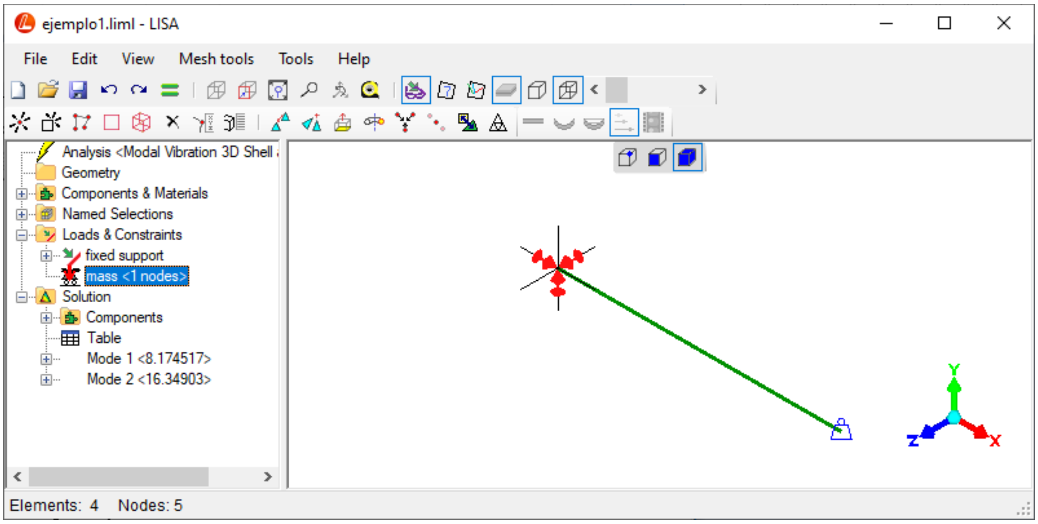

Figure IV.- Example 1, cantilever beam, FEM model in LISA 8

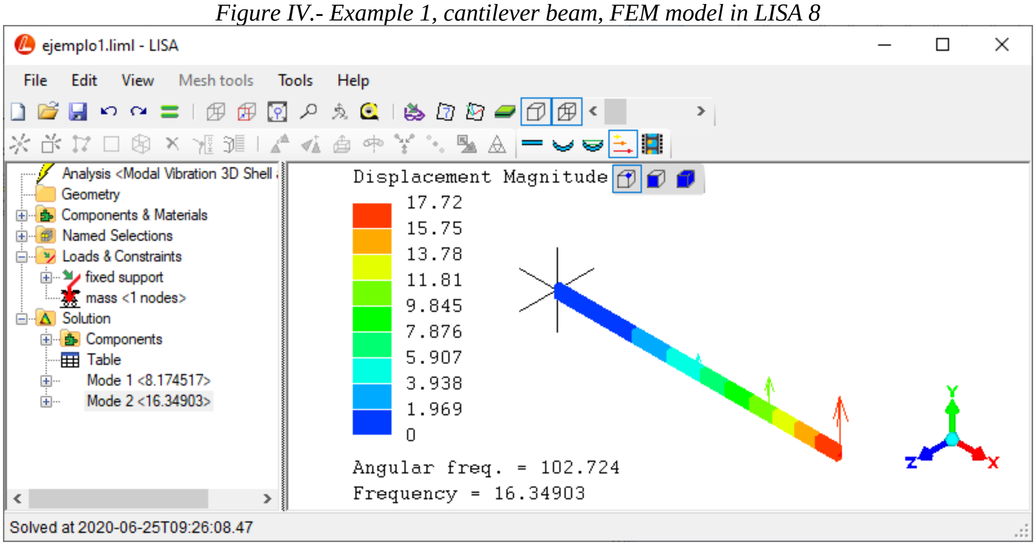

Figure V.- Example 1, results of the modal analysis of a cantilever beam in LISA 8

Loaded mass (kg)	Angular frequency (rad/s)
0.2	295
0.4	240
0.8	185
1	168
2	124
3	103
4	90

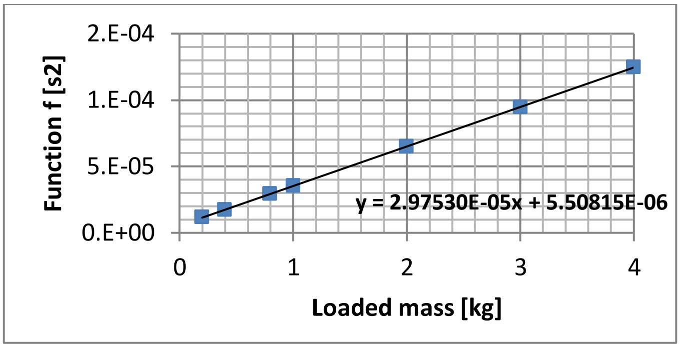

Table II & Figure VI- Example 1, angular frequencies and the "f" function versus the loaded mass, calculated with LISA 8.

The effective mass and the rigidity of the structure are found by replacing the slope and the constant of the linear approximation on equations. 18 and 19.

$$k = a^{-1} = 33612[N/m];\ m_{eff} = b.k = 0{,}1851\ [kg] \quad (20)$$

Finally, the ratio of the effective mass and the structural mass is:

$$r_{eff} = \frac{m_{effC}}{m_s} = 23{,}7\% \quad (21)$$

The table III shows the values of mass, the displacement of the modal analysis, the initial acceleration, and the inertial force for every node.

Node	Mass [kg]	Eigenvalue displacement	Initial acceleration [m/s^2]	Inertial force [N]
1	0.0975	0	0	0
2	0.195	1.858	0.53	0.1
3	0.195	6.733	1.91	0.37
4	0.195	13.58	3.85	0.75
5	3.0975	21.39	6.06	2.4

Table III.- Example I, values of inertial force the instantly applied weight load.

Finally, the stress analysis was performed on LISA of the dynamic problem using the static analysis model. Fig. VII shows the result of the longitudinal stress field.

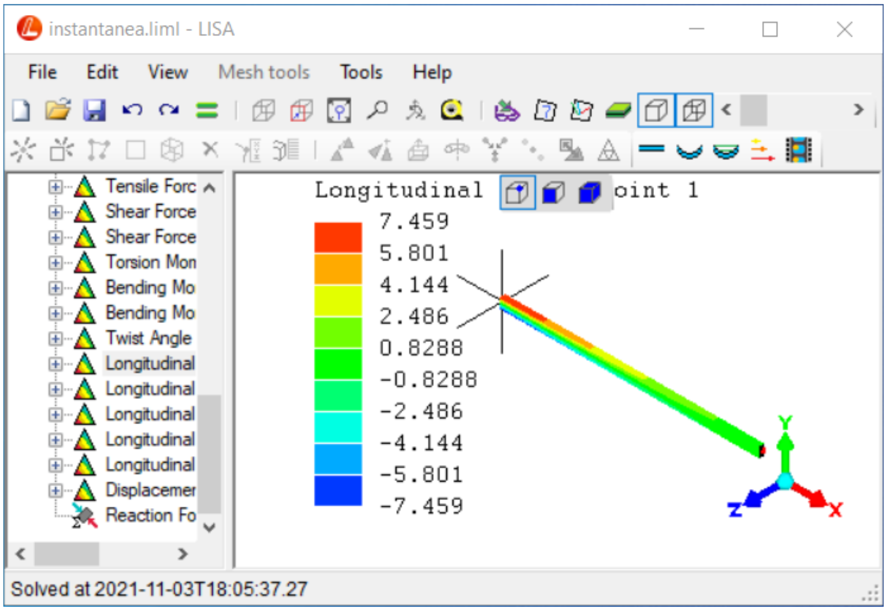

Figure VII- Example 1, Result of the stress analysis of the dynamic problem using the static analysis model.

The analytical Raleigh method [2] allows the verification. The rigidity and the effective mass of the Cantilever beam is given by

$$k = \frac{3EJ}{l^3} = 33600[N/m];\; m_{effA} = \frac{33}{140} m_s = 0.1838\ [kg] \tag{22}$$

The comparison of the effective mass calculated by analytical and numerical methods gives the following relative error (in percent):

$$e_{\%} = \frac{|m_{effA} - m_{effN}|}{m_{effA}} 100\% = 0.7\% \tag{23}$$

It must be remarked that this value is acceptable for many engineering applications.

3.2. Example 2.- Simply supported beam

This example presents a simply supported beam of rectangular cross section (width 50mm, height 80 mm) with a length of 2500 mm.

The model of the structure in LISA 8, with 20 beam elements and 21 nodes, has a simple support on each end (See fig. IIX). Seven runs were made with m_L values of 3, 9, 30, 50, 100, 150 and 200 kg placed on the beam center. The result of one run is shown in the screenshot in fig. IX. The results of the runs for the different values of m_L are shown in table IV, and the values of $f\,(m_L)$ are plotted in fig. X. The calculated slope is $7.252.10^{-7}$ [N/m], and the constant is $2.783.10^{-5}$ [s^2].

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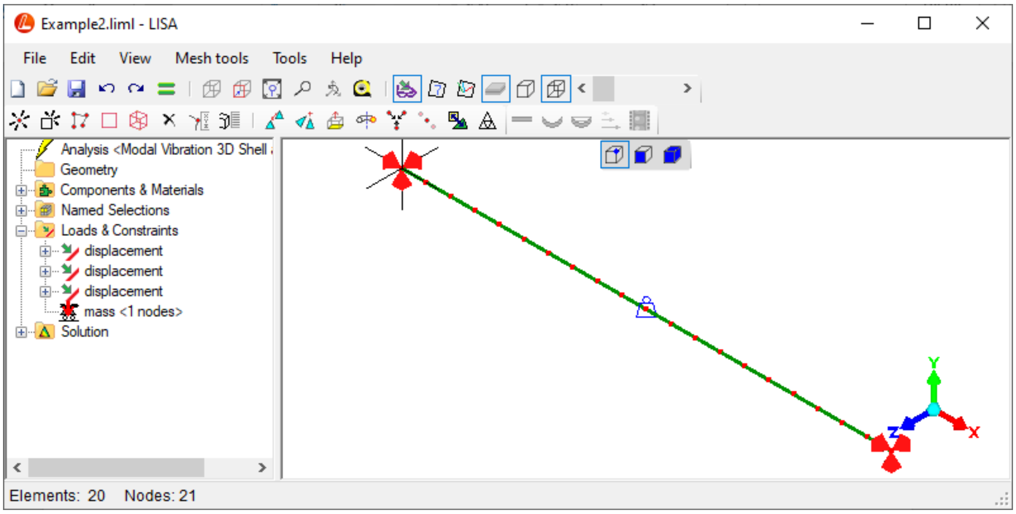

Figure IIX.- Example I1, simple supported beam, FEM model in LISA 8

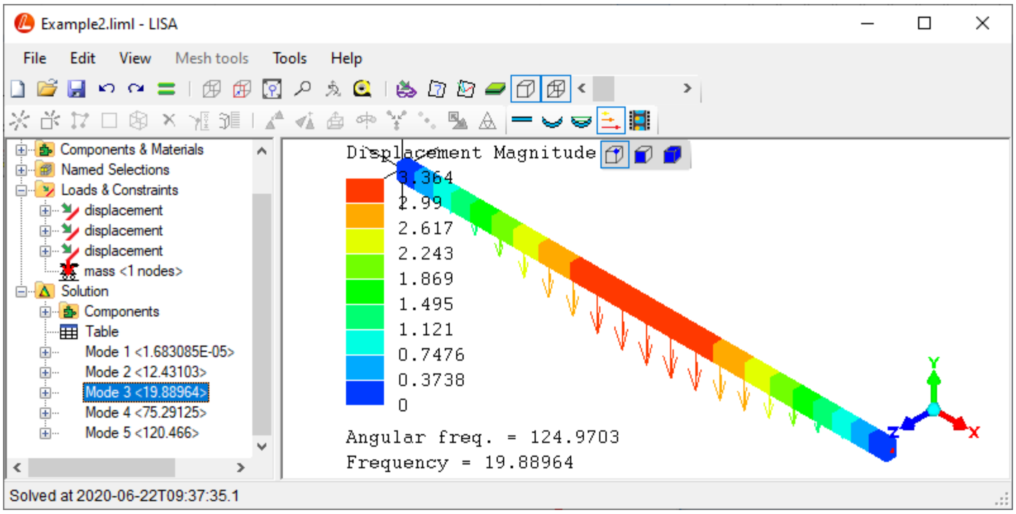

Figure IX.- Example I1, results of the modal analysis of a simple supported beam in LISA 8

Loaded mass (kg)	Angular frequency (rad/s)
3	182
9	171
30	142
50	125
100	100
150	86
200	76

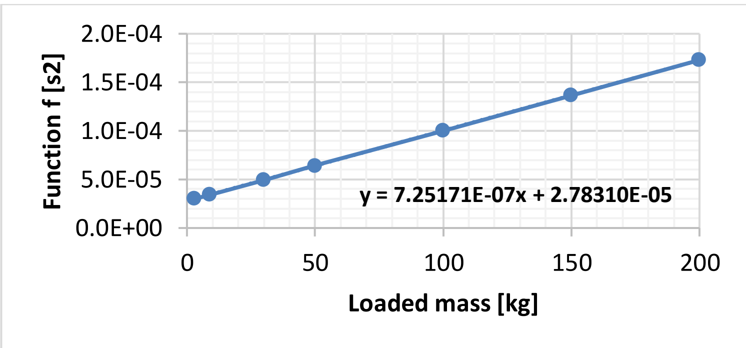

Table IV & Figure X- Example 2, natural angular frequencies of the structure and $f\,(m_L)$, calculated with LISA 8.

The effective mass and the rigidity of the structure are found by replacing, on eqs. 18 and 19, the slope and the constant of the linear approximation,

$$k = a^{-1} = 1379[N/m];\ m_{eff} = b.k = 38.38\ [kg] \tag{24}$$

Finally, the ratio of the effective mass to the structural mass is:

$$r_{eff} = \frac{m_{eff}}{m_s} = 49.20\% \tag{25}$$

For the verification, the analytical value of the natural frequency is obtained by the resolution of the Euler equation for a beam, assuming harmonic motion, and the simply support boundary conditions [3].

$$k = \frac{48EI}{l^3} = 1376[N/mm];\ \omega_n = \pi^2\sqrt{\frac{EI}{l^3 m_s}} \tag{26}$$

The effective mass of the single degree-of-freedom system is obtained by replacing eq. 26 on eq. 2.

$$m_{effA} = \frac{48}{\pi^4} m_s = 38{,}44[kg]. \tag{27}$$

And the analytic ratio of the effective mass to the structural one is:

$$\frac{m_{effA}}{m_s} = \frac{48}{\pi^4} \cong 49{,}28\% \tag{28}$$

The comparison of the effective mass calculated by analytical and numerical methods gives the following relative error (in percent):

$$e_{\%} = \frac{|m_{effA} - m_{effN}|}{m_{effA}} 100\% = 0{,}15\% \tag{29}$$

As in the previous example, this error value is acceptable for many engineering applications.

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3.3. Example 3.- A complex structure

The third example is a complex structure of beam elements, with no analytical solution. Fig. XI describes its geometry, and the nodes that have restricted displacements. The main beams are tubes of square cross section (220mm x 220mm) and a wall thickness of 15 mm. The others are tubes of square cross section (110 mm x 110mm) and a wall thickness of 12 mm. The height of the structure is 3 meters, and its width is 3.5 meters.

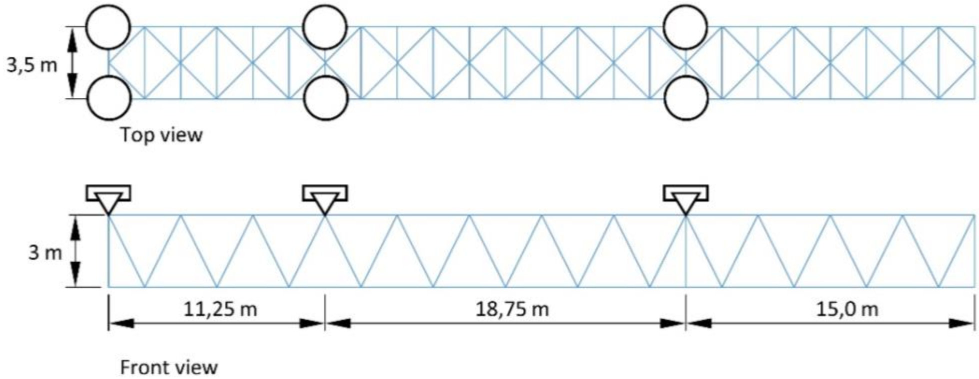

Figure XI.- Example I1I, geometry and boundary conditions.

The mesh was constructed with one element per beam, which is 173 elements and 69 nodes. Six runs were made with m_L values of 15, 35, 70, 105, 140 and 175 tons. The result of one run is shown on fig. XII. The results of the runs for the different values of m_L are shown in table V, and the values of $f\,(m_L)$ are plotted in fig. XIII. The calculated slope is 1,250.10^{-4} [N/mm], and the constant is 4,391.10^{-5} [s^2].

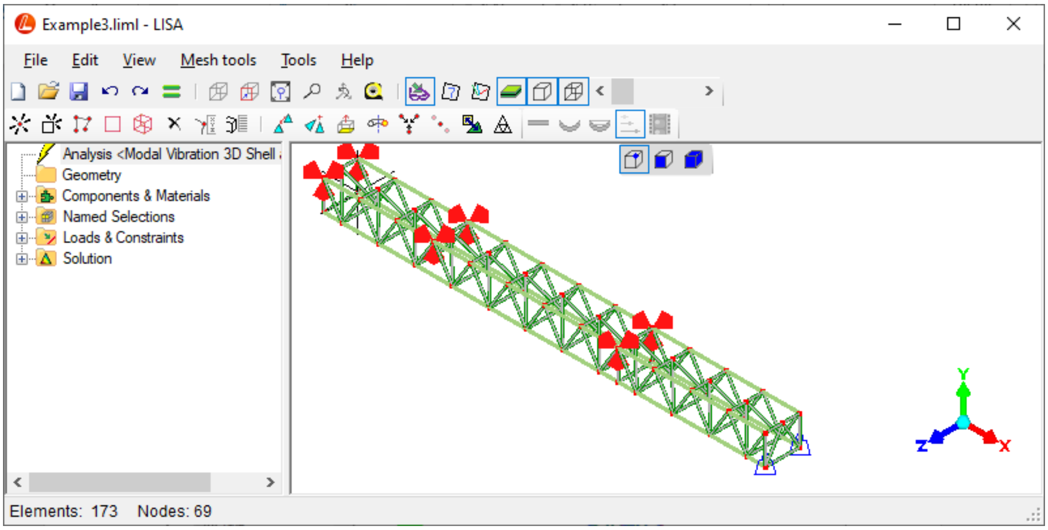

Figure XI.- Example II1, a complex beam structure, FEM model in LISA 8.

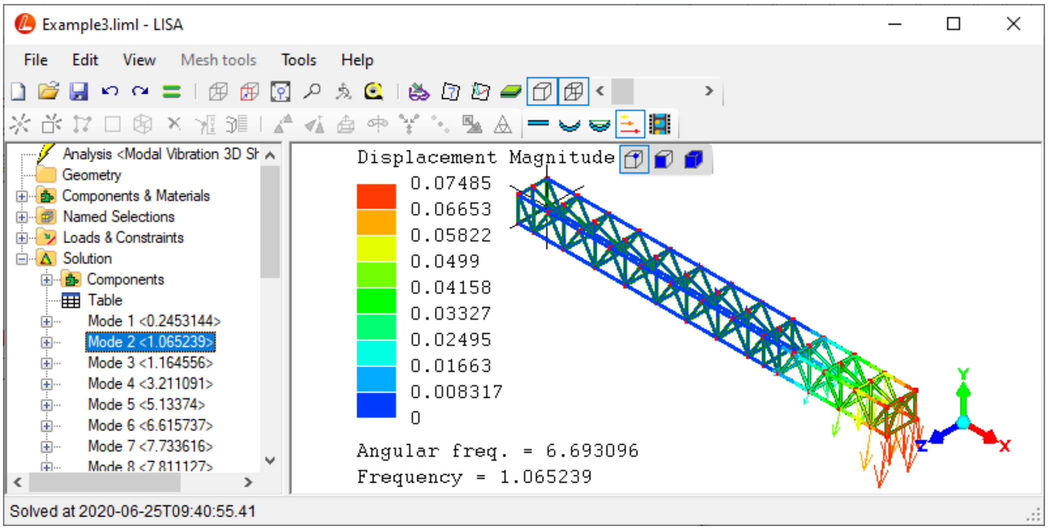

Figure XII.- Example II1, results of the modal analysis of a simple supported beam in LISA 8.

Loaded mass (ton)	**Angular frequency (rad/s)**
15	21
35	14
70	10
105	8,6
140	7,5
175	6,7

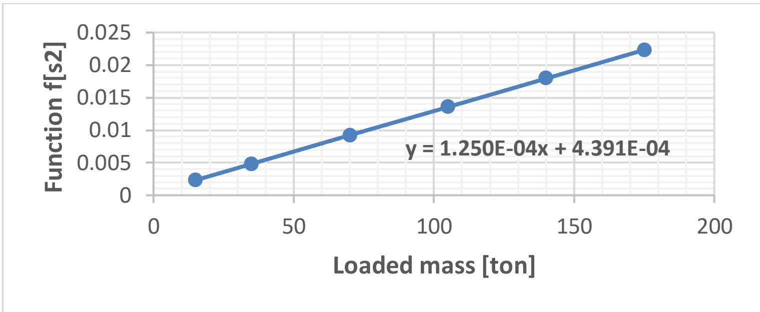

Table V.& Figure XIII- Example III, angular frequencies and $f\,(m_L)$ versus the loaded mass, calculated with LISA 8.

The effective mass and the rigidity of the structure are found by replacing the slope and the constant of the linear approximation, on eq. 18 and 19.

$$k = a^{-1} = 7997[N/mm];\ m_{eff} = b.k = 3{,}51\ [ton] \tag{30}$$

Finally, the ratio of the effective mass to the structural mass is:

$$r_{eff} = \frac{m_{eff}}{m_s} = 11{,}5\% \tag{31}$$

4. Conclusions. -

This work shows a method to obtain the effective mass and the stiffness of a structure modeled as a damped single degree-of-freedom system. The calculations require the natural angular frequencies of the structure modes, for different values of the loaded mass. The determination of the angular frequencies for different loaded masses can be obtained either experimentally or calculated by a modal analysis using the finite element method. Nowadays, the calculations may be carried out with a low-cost finite element software. This method is particularly useful for complex structures with no analytical solutions. The comparison of the results for structures with analytical solutions indicates that the errors are acceptable for many engineering applications (less than 1%).

Three examples show the capabilities of the method to perform the stress analysis, of the dynamic problem using a static analysis and the D'Alembert's Principle.

5. References

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Nota contribución de los autores:

1. Concepción y diseño del estudio

2. Adquisición de datos

3. Análisis de datos

4. Discusión de los resultados

5. Redacción del manuscrito

6. Aprobación de la versión final del manuscrito

FSS ha contribuido en: 1, 2, 3, 4, 5 y 6.

PS ha contribuido en: 1, 2, 3, 4, 5 y 6.

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